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HOW TO PRESENT THE HEART OF NEWTON'S PRINCIPIA TO THE LAYPERSON: A PRIMER ON THE CONIC SECTIONS WITHOUT APOLLONIUS OF PERGA

Takeshi Sugimoto

Applied Mathematician, (b. Tokyo, Japan, 1956).

Address: Faculty of Engineering, Kanagawa University, 3-27-1 Rokkakubashi, Kanagawa Ward, Yokohama, 221-8686, Japan E-mail: take@is.kanagawa-u.ac.jp

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Abstract: *Newton's Philosophiae Naturalis Principia Mathematica is written in Latin and deeply depends on Conics by Apollonius of Perga. Both are the hurdles for modern readers to wish to read Principia. Language problems may be solved by good translations. The aim of this study is to prepare a primer on the conic sections dependent only on the knowledge of the first year students in the tertiary education without using calculus or higher analytic geometry. The purpose is fulfilled by use of symmetry,* i.e.*, a circle and a rectangular hyperbola for elementary demonstrations, as well as coordinate straining to generalise the special results. In case of a parabola its symmetry is made most of, too. The result is structured by three definitions, two lemmas and six theorems to tackle at Newton's Lemmas XII, XIII, XIV and the Power of a Point Theorem.*

Keywords: Principia, conics, circle, rectangular hyperbola, coordinate straining.

1. INTRODUCTION

The most famous but least read book is Sir Isaac Newton's Philosophiae Naturalis Principia Mathematica, in short Principia hereinafter. The original is revised twice. The third edition was published in 1726. A year later Newton passed away. The text is written in Latin, but the language problem may be solved by good translations (for example Cohen & Whitman, 1999). The genuine difficulty lies in mathematics used to describe demonstrations in Principia. The celestial orbits are the conic sections, ellipses for the planets and comets and parabolas for some comets. Newton frequently states 'ex conicis', that is 'from Conics', and then he uses the results as the trivia. Modern readers, even scientists, lost the knowledge on Conics by Apollonius of Perga. Newton's manuscripts (Whiteside, 1967) reveal that he gained the knowledge on Conics by annotating Van Schooten (1657, 1659). These texts by Van Schooten are mixture of Descartes' analytic geometry and Apollonius' synthetic geometry. The history of studies in conic sections is concisely reviewed by Coolidge (1968).

The aim of this study is to prepare the basis of understanding the essential part of Principia without using Conics, calculus or higher analytic geometry. Our definition of the layperson is the people with the knowledge of the first year students in the tertiary education. We define the heart of Principia by Propositions VI, XI, XII and XIII in the Book I of Principia. These consist of demonstrations on the law of the universal gravitation.

We try to complement the former works by Brackenridge (1995), lacking explanations about Conics, and Matsumoto (1999), only with explanations about ellipses. Both afford us much inspiration, but they deal only with the elliptic case. We cover all the conic sections.

The strategy is to make most of symmetry and coordinate-staining: we demonstrate our lemmas and theorems by use of a circle and a rectangular hyperbola and generalise the results by straining the coordinate. In case of a parabola we also make most of its symmetry.

The structure of the paper is as follows: in the section entitled 'A primer on the conic sections' we concisely summarise the basic properties of the conic sections, demonstrate Newton's Lemmas XII, XIII, XIV and the Power of a Point Theorem; in the section entitled 'Reading the heart of Principia' we present the essentials, Newton's Propositions VI, XI, XII and XIII in Book I, for the layperson to understand 'the equation of the universe'; in the end we summarise our proposal.

2. A PRIMER ON THE CONIC SECTIONS

We use '$\sim$' for 'be similar to' and '$\cong$' for 'be congruent to.'

2.1. Ellipse

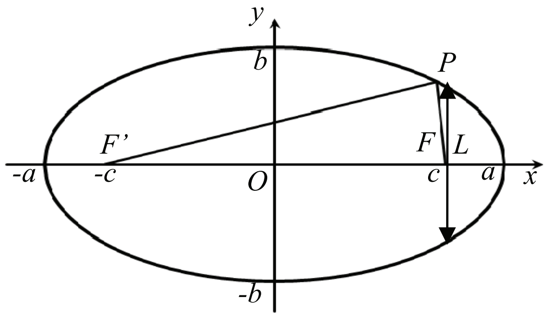


Fig.1. An ellipse.

Definition I: Ellipse.
An ellipse is the locus of all the points in a plane the sum of whose distances from two foci is constant. Using the elementary analytic geometry we derive the standard form of an ellipse as shown in Fig.1. Given an arbitrary point P at (x,y) on an ellipse with the major radius a and the minor radius b, we use the Pythagorean Theorem to obtain the distances of P from the focus F at $(c,0)$ and from F' at $(-c\,,0)$.

$$|PF| = \sqrt{(x-c)^2 + y^2}\,,$$

$$|PF'| = \sqrt{(x+c)^2 + y^2}\,.$$

If P is on the abscissa, the sum of the distances is equal to $2a$. Therefore

$$\sqrt{(x-c)^2 + y^2} + \sqrt{(x+c)^2 + y^2} = 2a.$$

Transposing the second term on the left-hand side to the right-hand side and squaring that equation twice, we obtain the standard form of an ellipse after some algebra.

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1. \tag{1}$$

To derive this we use the relation: $c^2 = a^2 - b^2$.

Since Newton uses the length of the latus rectum to derive the law of the gravitation, we need to find it. The principal latus rectum is the chord through a focus parallel to the minor axis as shown in Fig.1. Let L be the length of the principal latus rectum. Substituting c for x and $L/2$ for y in (1), we obtain

$$L = \frac{2b^2}{a}. \tag{2}$$

As a special case we choose a unit circle by substituting $a = b = 1$ and $c = 0$ in (1).

Theorem I: Newton's Lemma XII in Book I of Principia in case of ellipses
We shall quote Newton's Lemma from Motte's translation (1729):

> All parallelograms circumscribed about any conjugate diameters of a given ellipsis or hyperbola are equal among themselves.
>
> This is demonstrated by the writers of conic sections.

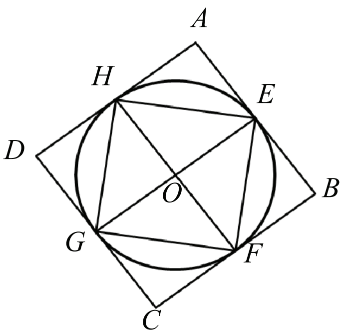
Proof.
We trace Matsumoto's demonstration (1999). To read Principia we need to introduce the conjugate diameters, each one bisecting all the chords parallel to the other. In case of a circle a pair of orthogonal diameters corresponds to the conjugate diameters.

We consider a pair of the orthogonal diameters EG and FH of a unit circle as shown in Fig.2 (a). Because of its symmetry it is obvious that all the chords parallel to FH are bisected by the diameter EG and *vice versa*. Therefore EG and FH are conjugate. We know tangents of a circle: tangents are lines touching a circle only once and perpendicular on the diameter. We draw four tangents to E, F, G and H. Let A, B, C and D be the intersections of the tangents. The four-edged figures $ABCD$ and $EFGH$ are squares, and their areas are 4 and 2, respectively. Therefore the areas of the squares defined by any set of the conjugate diameters are all the same in case of a circle.

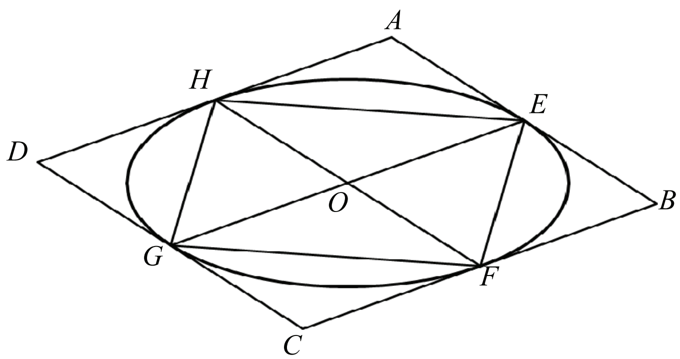
Straining the ordinate by the ratio b and the abscissa by the ratio a, respectively, we generalise the above-mentioned arguments for all the ellipses. Figure 2 (b) shows the result of a strained circle, *i.e.*, an ellipse. Since an area is a product of a height and a width, we should note that a unit area in Fig.2 (a) is magnified by ab for the corresponding strained area in Fig.2 (b).

Given an arbitrary point E on the ellipse, for example, we define the conjugate diameters as follows: a line from E through the centre is one diameter EG; the conjugate diameter FH is parallel to the tangents to E and G. We show how to draw the tangents to an ellipse in Theorem II below. In case of an ellipse the four-edged figures defined by a pair of the conjugate diameters are parallelograms $ABDC$ and $EFGH$, whose areas are $4ab$ and $2ab$ and independent of a choice of a pair of the conjugate diameters. Even if we choose the point E at any location on a unit circle, any areas are identical. Therefore the strained figures hold the same properties.

[Q.E.D.]



(a)



(b)

Fig.2. Tangents and conjugate diameters of ellipses: Newton's Lemma XII in Book I of Principia; (a) a circle and (b) an ellipse.

Theorem II: Tangents to an ellipse
Extend $F'P$ as far from P as F, and define another end point as G. Draw a line equidistant from F and G. This is the tangent to P on the ellipse.

Proof.
This is a well-known result. At any Q on this line the following

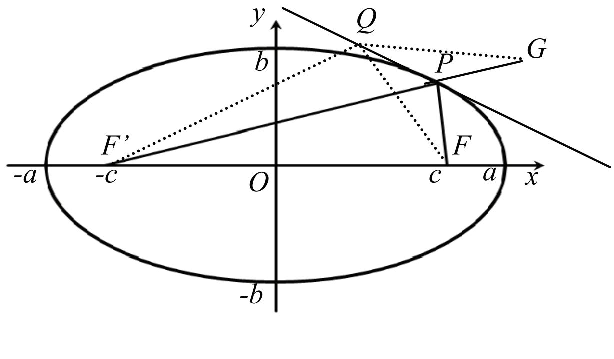


Fig.3. An ellipse and a tangent to it.

inequality holds.

$$F'Q + QF = F'Q + QG$$
$$\geq F'G$$
$$= F'P + PF$$

because of the Triangle Inequality applied to $\Delta F'QG$. This implies Q is not on the ellipse except $Q = P$. Therefore this line touches the ellipse at P just once.

The result also implies the angle of $F'P$ to the tangent is equal to the angle of FP to the tangent. [Q.E.D.]

Theorem III: Power of a Point Theorem for an ellipse

Let PG and DK be a pair of the conjugate diameters as shown in Fig.4. The chord QQ' is parallel to DK. Then the following holds.

$$Pv \cdot vG : Qv^2 = PC^2 : CD^2. \tag{3}$$

Newton writes PvG for $Pv \cdot vG$. Newton uses the relation (3) for an ellipse by stating 'ex conicis.'

Proof.

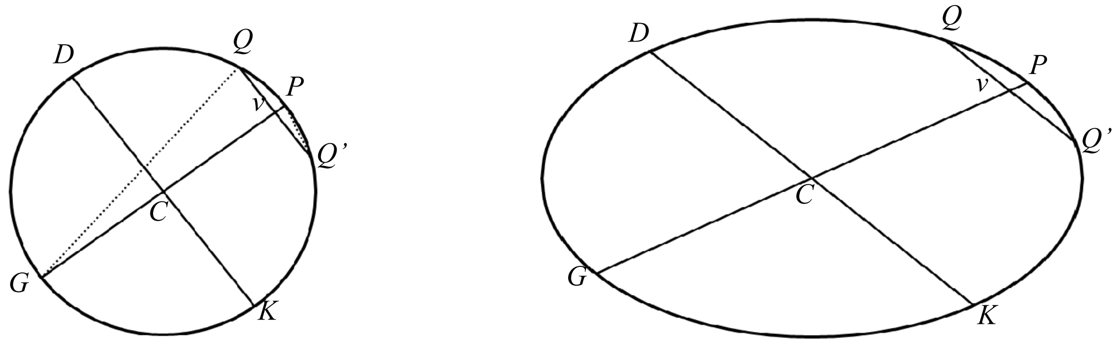
This is also Matsumoto's strategy (1999).

First we revisit the Power of a Point Theorem for a unit circle as shown in Fig.4 (a). The chord QQ' is bisected by PG at v. The inscribed angles $\angle GQQ'$ and $\angle GPQ'$ to the chord GQ' are the same, while another inscribed angles $\angle QGP$ and $\angle QQ'P$ to the chord QP are the same. Therefore $\Delta GQv \sim \Delta Q'Pv$ because of the AAA Theorem. Comparing the corresponding edges, we obtain the relation:

$$vG : Qv = Q'v : Pv,$$

or

$$Pv \cdot vG = Qv^2, \tag{4}$$



(a) (b)

Fig.4. Power of a Point Theorem for ellipses:
(a) a unit circle; (b) an ellipse.

because $Qv = Q'v$. If QQ' coincide with DK, the following holds.

$$PC^2 = CD^2. \tag{5}$$

Making the product of the left-hand side of (3) and the right-hand side of (4) as well as the product of the right-hand side of (3) and the left-hand side of (4), we obtain

$$Pv \cdot vG \cdot CD^2 = Qv^2 \cdot PC^2. \tag{6}$$

Therefore (3) holds in case of a unit circle.

Straining the ordinate by the ratio b and the abscissa by the ratio a, respectively, we generalise the theorem to all the ellipses shown in Fig.4 (b). Suppose the conjugate diameters PG and DK are re-scaled by the coordinate-straining to α-times and β-times the original lengths, respectively, then the both-hand sides of (6) are multiplied by $\alpha^2\beta^2$. Therefore the relation (6) holds true to any ellipses. [Q.E.D.]

2.2. Hyperbola

Definition II: Hyperbola.

A hyperbola is the locus of all the points in a plane the difference of whose distances from two foci is constant. Using the elementary analytic geometry we derive the standard form of a hyperbola as shown in Fig.5. Given an arbitrary point P at (x,y) on a hyperbola with the x-intercepts $\pm a$, we use the Pythagorean Theorem to obtain the distances of P from the focus F at $(c,0)$ and from F' at $(-c\,,0)$.

$$|PF| = \sqrt{(x-c)^2 + y^2},$$

$$|PF'| = \sqrt{(x+c)^2 + y^2}.$$

If P is on the abscissa, the difference of the distances is equal to $2a$. Therefore

$$\sqrt{(x+c)^2 + y^2} - \sqrt{(x-c)^2 + y^2} = 2a.$$

Transposing the second term on the left-hand side to the right-hand side and squaring that equation twice, we obtain the standard form of an ellipse after some algebra.

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = 1. \tag{7}$$

To derive this we use the relation: $c^2 = a^2 + b^2$.

In the very same manner another curves with the y-intercepts $\pm b$ is given by

$$\left(\frac{x}{a}\right)^2 - \left(\frac{y}{b}\right)^2 = -1. \tag{8}$$

Diagonal lines in Fig.4 are asymptotes given by

$$\frac{x}{a} \pm \frac{y}{b} = 0.$$

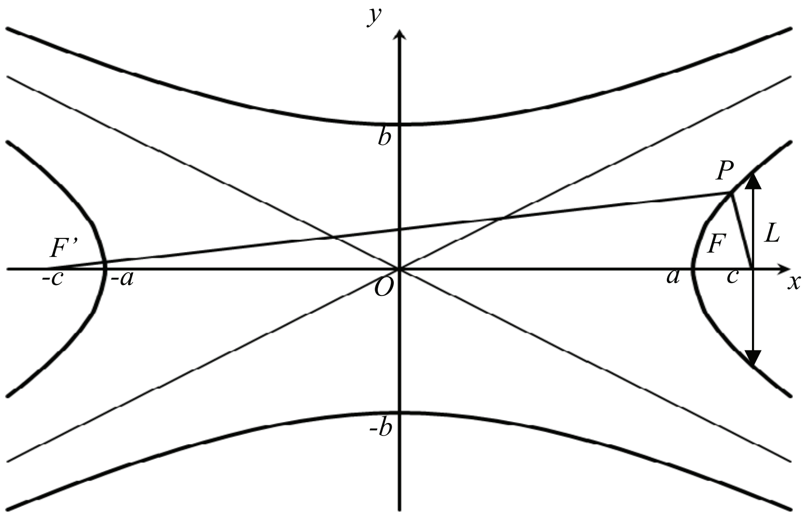


Fig.5. A hyperbola.

We need to find the length of the latus rectum, L, as shown in Fig.5. Substituting c for x and $L/2$ for y in (7), we obtain

$$L = \frac{2b^2}{a}. \qquad (9)$$

As a special case we choose a rectangular hyperbola by setting $a = b = 1$ and $c^2 = 2$. That is

$$x^2 - y^2 = 1. \qquad (10)$$

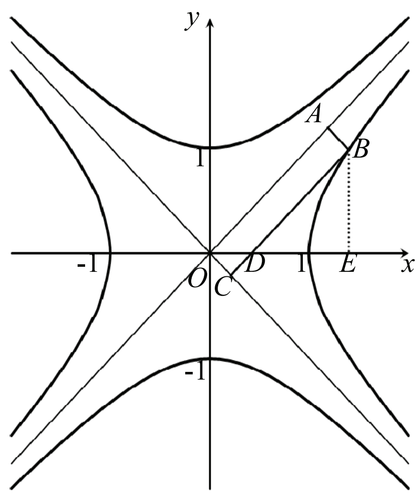


Fig.6. A rectangular hyperbola.

This is shown in Fig.6. Let us confirm a rectangular hyperbola inscribes a rectangle. For the brevity we consider a quarter of the inscribed rectangle $ABCO$ as shown in Fig.6. The segments AB and BC are parallel to the corresponding asymptotes. Noting the asymptotes for a rectangular hyperbola are given by $y = \pm x$, we find the slopes of the asymptotes are ±45 degrees. The point D is an intersection of BC and the abscissa, while the point E is a foot of the perpendicular from B to the abscissa. The triangle ΔBDE is an isosceles right triangle, because $\angle BDE$ is 45 degrees. Therefore $BE = DE$. The triangle ΔDOC is also an isosceles right triangle, because $\angle DOC$ is –45 degrees. Therefore $DC = OC$. Let (x,y) be the coordinate of the point B. Since $BE = y$, we apply the Pythagorean Theorem to obtain

$$BD = \sqrt{2}y.$$

Since $OD = x–y$, we apply the Pythagorean Theorem to obtain

$$\begin{aligned} OC &= DC \\ &= \frac{x-y}{\sqrt{2}}. \end{aligned}$$

Since $AB = OC$, the area of the rectangle $ABCO$ is given by

$$\begin{aligned} AB \cdot BC &= \frac{x-y}{\sqrt{2}}\left(\sqrt{2}y + \frac{x-y}{\sqrt{2}}\right) \\ &= \frac{x-y}{\sqrt{2}}\frac{x+y}{\sqrt{2}} \\ &= \frac{x^2-y^2}{2} \\ &= \frac{1}{2}, \end{aligned}$$

because of (10). Therefore the area of the inscribed rectangle is all the same in case of a rectangular hyperbola.

Lemma I: Three similar rectangles and parallel lines

Let us consider a rectangle $ABCD$ partitioned by the segments EG and FH parallel to BC and AB, respectively, as shown in Fig.7.

If the area of the rectangle $AEIH$ is equal to that of the rectangle $FCGI$, then

$$EF \,//\, AC.$$

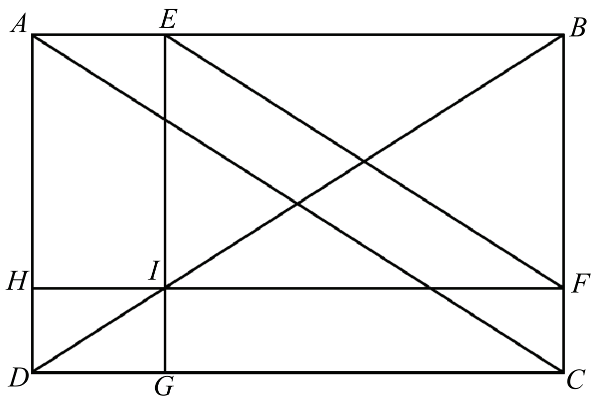


Fig.7. Three similar rectangles and parallel lines.

Proof.

Because of the assumption

$$AE \cdot EI = IF \cdot FC,$$

or

$$AE : FC = IF : EI.$$

Since $AE = HI$ as well as $FC = IG$,

$$HI : IG = IF : EI.$$

Therefore the rectangle $HIGD$ is similar to the rectangle $EBFI$; hence the points B, I and D are on the single segment; that is the rectangle $ABCD$ is also similar to the rectangles

HIGD and *EBFI*. Then their diagonals hold the relation:

EF // *AC*.

[Q.E.D.]

Corollary.

We are given arbitrary points *P* and *Q* on a rectangular hyperbola as shown in Fig.8. Let us draw perpendiculars from *P* and *Q* to the nearby asymptotes. The points *P'* and *Q'* are the feet of these perpendiculars, respectively, while *P"* and *Q"* are mirror images of *P* and *Q* about the corresponding asymptotes, respectively.

Then

PQ // *P'Q'* // *P"Q"*,

because of the identity of the areas of the inscribed rectangles in a rectangular hyperbola and Lemma I.

[Q.E.D.]

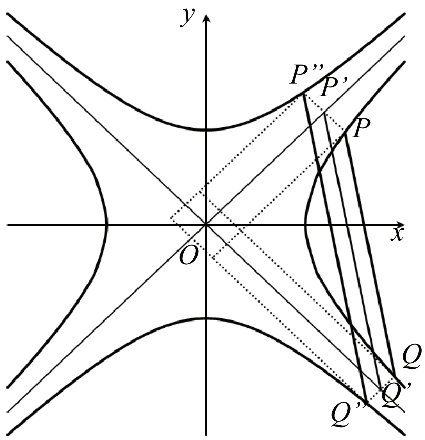


Fig.8. Corollary of Lemma I.

Theorem IV: Tangents and conjugate diameters in a rectangular hyperbola

Given an arbitrary point *P*, we define the tangents and conjugate diameters by use of diagonals *AB* and *PC* of a quarter of the inscribed rectangles *APBC* in a rectangular hyperbola as shown in Fig.9.

The tangent to *P* is a line passing through *P* and parallel to *AB*.

The segments passing through the centre *C* and parallel to *PC* and *AB*, i.e., *PG* and *DK*, constitute a pair of the conjugate diameters.

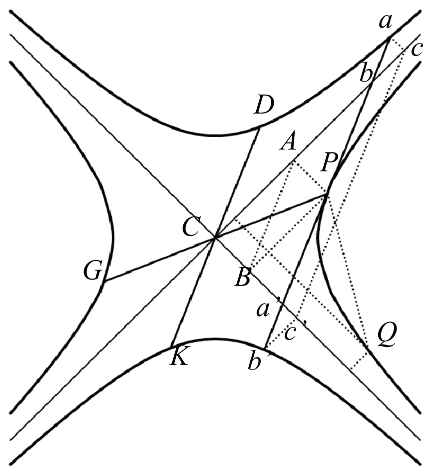


Fig.9. Tangents and conjugate diameters in a rectangular hyperbola.

Proof.

If a line intersects the curve at *P* and *Q*, the segment *PQ* joins the corners of two inscribed rectangles. Therefore the slope of *PQ* differs from that of the diagonal *AB*; two slopes coincide with each other, if and only if *Q* is at *P*. Therefore the line through *P* and parallel to *AB* does not intersect the touching curve any more.

The tangent to *P* intersects the nearby asymptotes at *b* and *a'*, while the same tangent intersects other curves of a hyperbola at *a* and *b'*. Let *c* and *c'* be the feet of the perpendiculars from *a* and *b'* to the nearby asymptotes, respectively. We shall show *ab'* is bisected at *P*.

First we show three triangle ΔABC, ΔbAP and $\Delta PBa'$ are congruent. Since AB // aPa', $\angle CBA = \angle APb$ as well as $\angle CAB = \angle BPa'$; $\angle ACB$, $\angle bAP$ and $\angle PBa'$ are all right angles; $CB = AP$ as well as $AC = PB$; hence $\Delta ABC \cong \Delta bAP$ as well as $\Delta ABC \cong \Delta PBa'$ because of the ASA Theorem; therefore $bP = Pa'$.

Next we show two triangles Δabc and $\Delta a'b'c'$ are congruent. The tangent ab' // cc' because of Corollary of Lemma I; hence $ac = a'c'$ as well as $\angle bac = \angle b'a'c'$; $\angle acb = \angle a'c'b'$ are all right angles; hence $\Delta abc \cong \Delta a'b'c'$; therefore $ab = a'b'$.

Collecting those results $aP = Pb'$.

In the very same manner we find any chords parallel to *AB* is bisected by *PG*.

If the segment *DK* is parallel to *AB*, then *DK* bisects any chords parallel to *PG* by use of the same argument above.

Since each one bisecting all the chords parallel to the other, *PG* and *DK* constitute a pair of the conjugate diameters. [Q.E.D.]

Theorem V: Newton's Lemma XII in Book I of Principia in case of hyperbolas

We repeat to quote Newton's Lemma from Motte's translation (1729):

> All parallelograms circumscribed about any conjugate diameters of a given ellipsis or hyperbola are equal among themselves.
>
> This is demonstrated by the writers of conic sections.

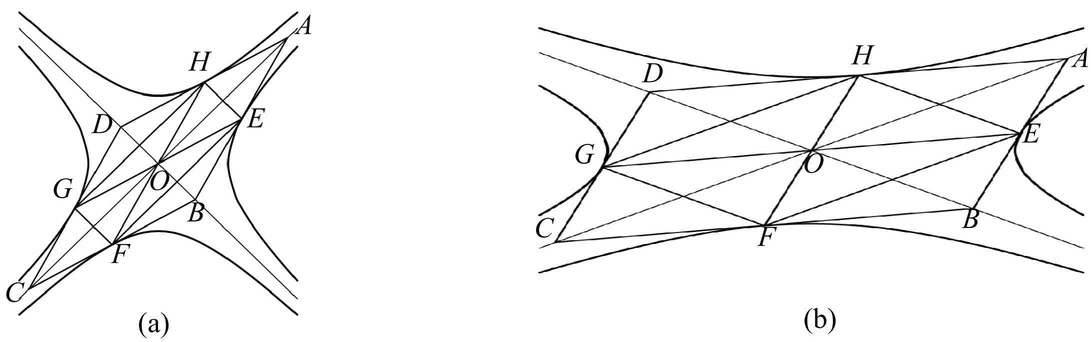


Fig.10. Tangents and conjugate diameters in hyperbolas: Newton's Lemma XII in Book I of Principia; (a) a rectangular hyperbola; (b) a hyperbola.

Proof.
In Definition II we find the identity of the areas of the inscribed rectangles in a rectangular hyperbola. The areas of the parallelogram *ABCD* and the rectangle *EFGH* in Fig.10 (a) are 4 and 2, respectively.

Straining the ordinate by the ratio b and the abscissa by the ratio a, respectively, we generalise the above-mentioned area identity to all the hyperbolas.

Given an arbitrary point E, for example, we define the conjugate diameters as follows: a line from E through the centre is one diameter EG; the conjugate diameter FH is parallel to the tangents to E and G. We show how to draw the tangents to a hyperbola in Theorem VI below. In case of a general hyperbola the four-edged figures defined by a pair of the conjugate diameters are parallelograms *ABDC* and *EFGH*, whose areas are $4ab$ and $2ab$ and independent of a choice of a pair of the conjugate diameters. Even if we choose the point E at any location on a rectangular hyperbola, any areas are identical. Therefore the strained figures hold the same properties.

[Q.E.D.]

Theorem VI: Tangents to a hyperbola
Define the point G on $F'P$ as far from P as F. Draw a line equidistant from F and G. This is the tangent to P on the hyperbola.

Proof.
This is a well-known result. At any Q on this line the following inequality holds.

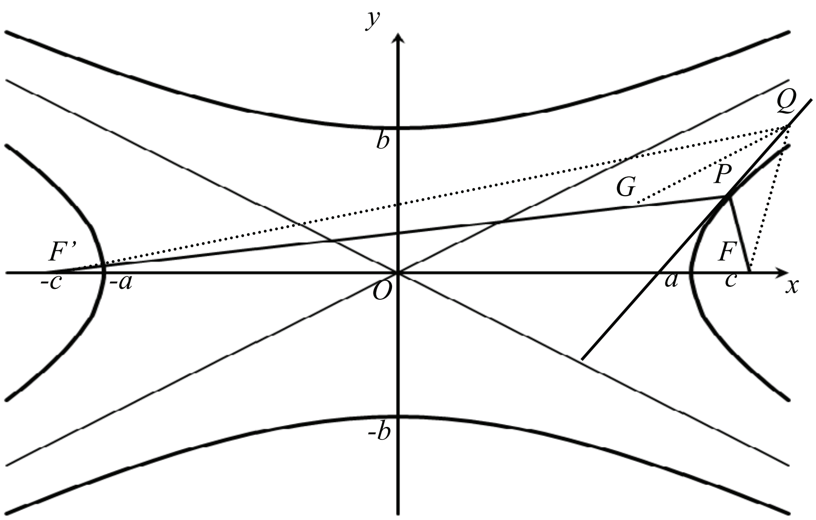


Fig.11. A hyperbola and a tangent to it.

$$F'Q - QF = F'Q - QG$$
$$\le F'G$$
$$= F'P - PF$$

because of the Triangle Inequality applied to $\Delta F'QG$. This implies Q is not on the curve except $Q = P$. Therefore this line touches the curve at P just once.

The result also implies the tangent to P is a bisector of $\angle F'PF$.

[Q.E.D.]

Theorem VII: Power of a Point Theorem for a hyperbola
Let PG, DK, C, QQ' and v be a pair of the conjugate diameters, the centre, the chord parallel to DK and the mid-point of QQ' as shown in Figs.12 and 13. Then the following holds.

$$Pv \cdot vG : Qv^2 = PC^2 : CD^2. \tag{3}$$

Newton writes PvG for $Pv \cdot vG$. Newton uses the relation (3) for a hyperbola by stating 'ex natura conicorum (by the properties of the conic sections).'

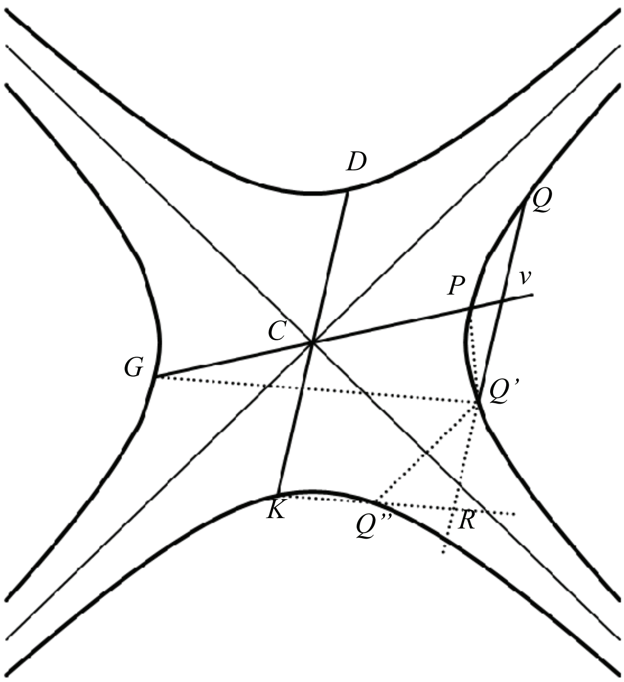


Fig.12. Power of a Point Theorem for a rectangular hyperbola.

Proof.
We shall demonstrate the proposition above by showing $\Delta GvQ' \sim \Delta Q'vP$ in case of a rectangular hyperbola.

We draw several auxiliary lines: join P and Q'; join G and Q'; draw a line from Q' to join its mirror image Q'' about the asymptote; join K and Q''; let R be the intersection of the extensions of QQ' and KQ''.

First we confirm that D to P, P to K, K to G and G to D are all mirror images about the corresponding asymptotes, respectively.

Next we confirm the following lines are parallel: DK // QR; GQ' //KR because of Corollary of Lemma I.

Because of the symmetry of the pentagon $CPQ'Q''K$, $\angle CPQ' = \angle CKQ''$; since DK // QR, $\angle CKQ''$ is a supplementary angle to $\angle Q'RQ''$; $\angle CPQ'$ is a supplementary angle to $\angle vPQ'$; hence $\angle vPQ' = \angle Q'RQ''$; $\angle GQ'v = \angle Q''RQ'$ because of the corresponding angles at the parallel lines GQ' and KR; hence $\angle vPQ' = \angle GQ'v$; $\Delta GvQ'$ and $\Delta Q'vP$ have $\angle PvQ'$ in common; therefore $\Delta GvQ' \sim \Delta Q'vP$ because of the AAA Theorem. Making the ratios of the corresponding edges in these triangles, we obtain.

$$vG : Q'v = Q'v : Pv$$

or we regain the equation (4)

$$Pv \cdot vG = Qv^2, \tag{4}$$

because $Qv = Q'v$. If QQ' coincide with DK, we regain the equation (5).

$$PC^2 = CD^2. \tag{5}$$

Making the product of the left-hand side of (3) and the right-hand side of (4) as well as the product of the right-hand side of (3) and the left-hand side of (4), we regain the equation (6).

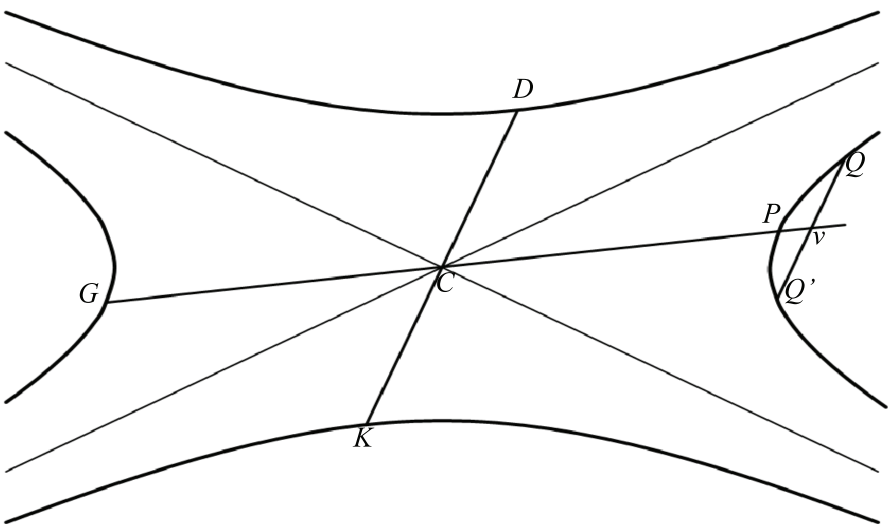


Fig.13. Power of a Point Theorem for a hyperbola.

$$Pv \cdot vG \cdot CD^2 = Qv^2 \cdot PC^2. \tag{6}$$

Therefore (3) holds in case of a rectangular hyperbola.

Straining the ordinate by the ratio b and the abscissa by the ratio a, respectively, we generalise the theorem to all the hyperbola shown in Fig.13 by use of the same argument made for an ellipse.

[Q.E.D.]

2.3. Parabola

Definition III: Parabola.

A parabola is the set of all the points in a plane equidistant from the focus and the directrix. Using the elementary analytic geometry we derive the standard form of a parabola as shown in Fig.14. Given an arbitrary point P at (x,y) on a parabola, we use the Pythagorean Theorem to obtain the distances from the focus F at $x = p$ and the directrix, the parallel line to the ordinate at the distance p, to P.

$$|PF| = \sqrt{(x-p)^2 + y^2},$$

$$|PH| = x + p,$$

and hence

$$\sqrt{(x-p)^2 + y^2} = x + p.$$

Squaring the equation above once, we obtain the standard form of a parabola after a little algebra.

$$y^2 = 4px. \tag{11}$$

Next we calculate the length of the principal latus rectum by substituting p for x and $L/2$ for y in (11). The result is as follows.

$$L = 4p. \tag{12}$$

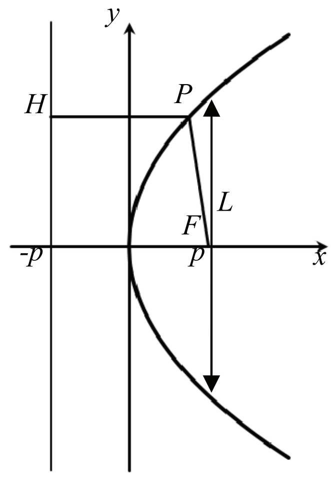


Fig.14. A parabola.

Theorem VIII: Tangents and their properties in a parabola

We are given an arbitrary point P at (x_p,y_p) on a parabola, $y^2 = 4px$, as shown in Fig.15 (a); let F and H be the focus and the foot of the perpendicular from P to the directrix.

Then the tangent to P is a bisector of an angel $\angle HPF$, and the slope of the tangent to P is given by $2p/y_p$.

Proof.

The former part of this proposition is a well-know result. Let Q and H' be the point on the tangent and the foot of the perpendicular from Q to the directrix, respectively, as shown in Fig.15 (a).

If we introduce a bisector of $\angle HPF$, then $QF = QH$; the following inequality holds due to the Triangle Inequality applied to $\Delta QHH'$: $QH' < QH$; hence $QH' < QF$. Therefore $PF = PH$ only at P. This implies this bisector is the tangent without intersecting the touching parabola any more.

We use Fig.15 (b) to calculate the slope of the tangent to P. Draw an auxiliary line to join H and F; let T be the intersection of the tangent and HF; draw a perpendicular from F to the extension of PH; let G be its foot.

The abscissa is parallel to HP, and hence the slope of the tangent is measured by $\angle HPT$; since $PF = PH$ as well as $\angle HPT = \angle FPT$, $PT \perp HF$; $\Delta HPT \sim \Delta HFG$ due to AAA Theorem, because either $\angle HTP$ or $\angle HGF$ is the right angle as well as $\angle PHT$ in common; hence $\angle HPT = \angle HFG$. Therefore the slope of the tangent is measured by $\angle HFG$, that is $|HG|/|GF|$. Substituting $2p$ for $|HG|$ and y_p for $|GF|$, we find the slope is given by $2p/y_p$.

[Q.E.D.]

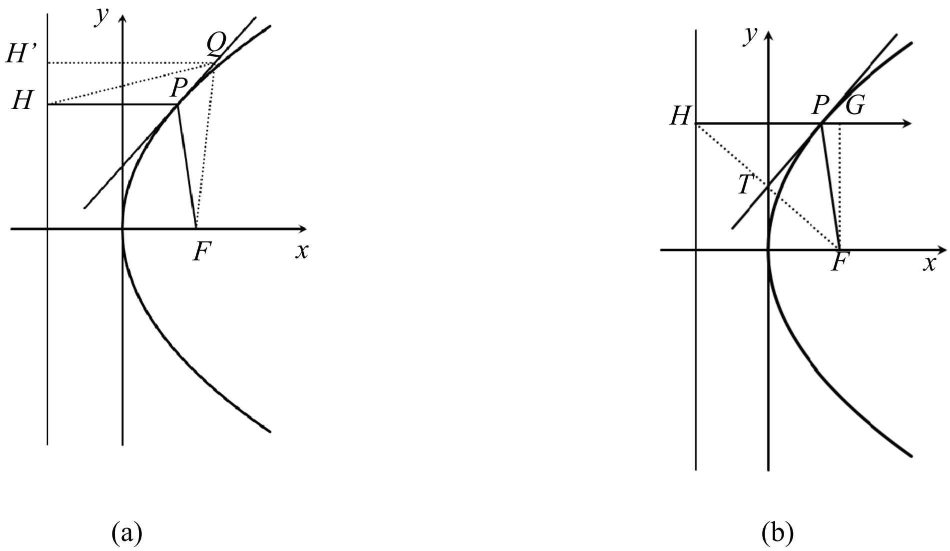


Fig.15. Tangents and their properties in a parabola:
(a) the definition; (b) the slope of the tangent to P.

Theorem IX: Newton's Lemma XIV in Book I of Principia

We shall quote Newton's Lemma XIV in Book I of Principia from Motte's translation (1729):

> The perpendicular let fall from the focus of a parabola on its tangent, is a mean proportional between the distances of the focus from the point of contact, and from the principal vertex of the figure.

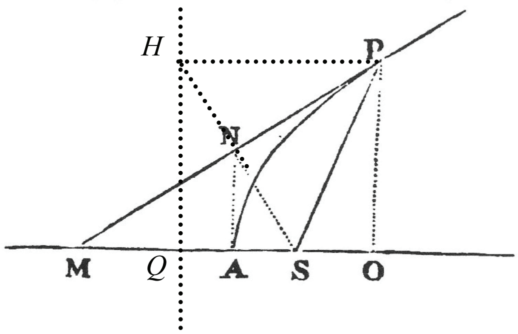


Fig.16. Newton's Lemma XIV in Book I of Principia: the figure is modified*) from Principia, the third edition (1726).

Proof.
Let A, S, P and H be the principal vertex, the focus, the point of contact and the foot of the perpendicular from P to the directrix, respectively; let M, N, O and Q be the intersection of the tangent and the abscissa, the foot of the perpendicular from S to the tangent, the foot of the perpendicular from P to the abscissa and the intersection of the directrix and the abscissa, respectively*).

The foot of the perpendicular from H to the tangent is also N, because PN bisects the isosceles triangle ΔHPS; $HQ \,//\, NA$ due to the Triangle Midpoint Theorem, because $HN = NS$ as well as $QA = AS$; hence $\angle NAS$ is the right angle; ΔPSM is an isosceles triangle, because $\angle SPN = \angle HPN = \angle SMN$; $\Delta MSN \cong \Delta PSN$, because SN bisects the isosceles triangle ΔPSM.

Therefore $\Delta NSA \sim \Delta MSN$ due to the AAA Theorem, because $\angle SNM = \angle SAN$ as well as $\angle NSA$ in common. Along with the congruence $\Delta MSN \cong \Delta PSN$, we obtain the ratios of the relevant edges of ΔNSA, ΔMSN and ΔPSN.

$$PS : SN = SN : SA, \tag{13}$$

or

$$SN^2 = PS \cdot SA.$$

Furthermore the repeated use of the relation above leads us to the Corollary I of Lemma XIV in Book I of Newton's Principia:

$$\begin{aligned} PS^2 : SN^2 &= PS^2 : PS \cdot SA \\ &= PS : SA \end{aligned} \tag{14}$$

This is used to solve Proposition XIII Problem VIII in Book I of Newton's Principia.

[Q.E.D.]

---

*) The points H and Q are added along with the directrix to Newton's original figure, and we do not use the point O. Newton uses the relation $MA = AO$, but we need to introduce H and Q to demonstrate that.

Lemma II: Diameters in a parabola

We are given an arbitrary point P at (x_p,y_p) on a parabola, $y^2 = 4px$. The line starting from P and parallel to the abscissa is a diameter, which bisects any chord parallel to the tangent to P.

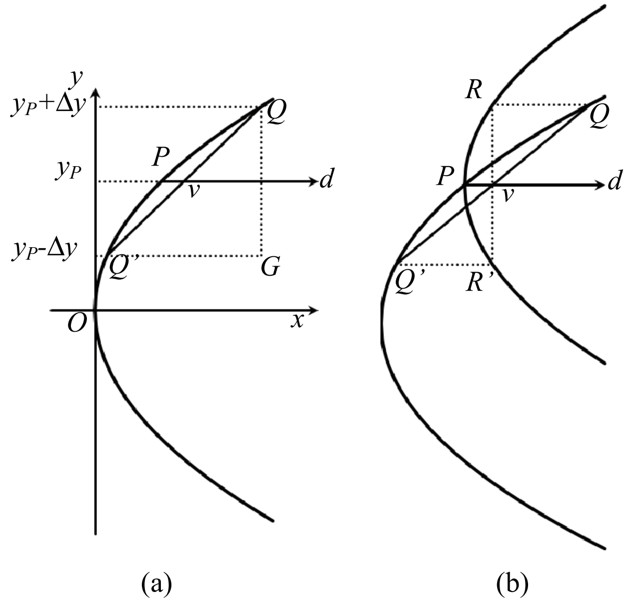


Fig.17. Ordinates, abscissas and diameters in a parabola: (a) the slope of the chord; (b) Pv vs. Qv.

Proof.

Let Q, Q', d and G be the points on the parabola that are equidistant along the ordinate above and below the point P, the horizontal line from P and the intersection of the horizontal line from Q' and the perpendicular from Q, respectively, as shown in Fig.17 (a).

Due to a counter-proposition of the Triangle Midpoint Theorem $Qv = Q'v$, because the line d parallel to $Q'G$ bisects QG.

Let us calculate the slope of the chord QQ' by use of the formula $QG/Q'G$.

$QG = 2\Delta y$,

and

$$Q'G = \frac{(y_p + \Delta y)^2}{4p} - \frac{(y_p - \Delta y)^2}{4p}$$

$$= \frac{\Delta y y_p}{p}.$$

Therefore the slope of the chord is found to be $2p/y_p$, which is equal to the slope of the tangent to P (Theorem VIII) and independent of Δy. This implies d is the diameter that bisects any chords parallel to the tangent to P.

Next we shall show Qv^2/Pv is constant by use of Fig.17 (b). We use the same points Q and Q' as Fig.15 (a). We introduce the congruent parabola that has P as a principle vertex. Draw the horizontal lines from Q and Q', and these parallel lines intersect the new parabola at R and R'.

Since RQ // $Q'R'$, QQ' and RR' have the midpoint v in common. The segment RvR' is perpendicular to the diameter d, because R and R' are on the new parabola symmetric about d. Hence any ΔRvQ and $\Delta R'vQ'$ are all similar at any v due to the AAA Theorem, because $\angle vQR$ and $\angle QRV$ are respectively identical at any v. Therefore the following relation holds.

Qv/Rv = const.

while on the parabola RPR' the following holds.

Rv^2/Pv = const.

Therefore

$$Qv^2/Pv = \text{const.} \tag{15}$$

[Q.E.D.]

Theorem VII: Power of a Point Theorem for a parabola

This is Newton's Lemma XIII in Book I of Principia. We shall quote Newton's Lemma from Motte's translation (1729):

> The latus rectum of a parabola belonging to any vertex is quadruple the distance of that vertex from the focus of the figure.
>
> This is demonstrated by the writers of the conic sections.

Proof.

We demonstrate the theorem by setting the chord QQ' running through the focus S. The directrix is shown far left in Fig.18. Let L be the length of the latus rectum.

$$\begin{aligned} L &= QQ' \\ &= QS + Q'S \\ &= QH' + Q'H''. \end{aligned}$$

Since the tangent to P is parallel to the chord QQ', $\angle PSv = \angle PvS$; hence $Pv = SP$ or $Hv = 2SP$; Hv is an arithmetic mean of QH' and $Q'H''$ due to a counterproposition of the Triangle Midpoint Theorem, because QH' // Hv // $Q'H''$ as well as $Qv = Q'v$. Therefore

$$\begin{aligned} 2SP &= Hv \\ &= \frac{QH' + Q'H''}{2} \\ &= \frac{L}{2}. \end{aligned}$$

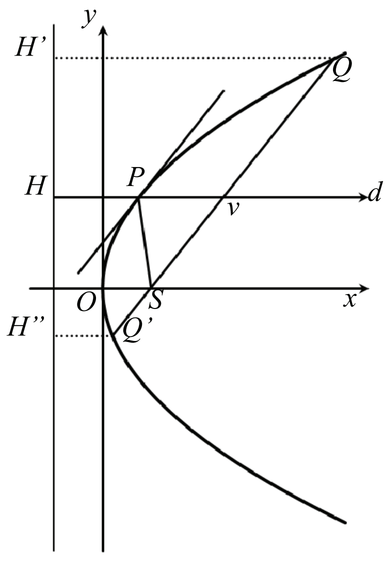


Fig.18. Newton's Lemma XIII in Book I of Principia.

That is

$$L = 4SP. \tag{16}$$

Substituting $2SP$ for Qv and SP for Pv in (15), we find the constant is $4SP$. The equation (15) is explicitly given by

$$Qv^2 = 4SP \times Pv. \tag{17}$$

The standard form (11) with (12) is found to be a special case of (17) with (16).

[Q.E.D.]

3. READING THE HEART OF PRINCIPIA

Now that we finish the preparation, we shall read the heart of Newton's Principia.

3.1. Proposition VI Theorem V and its corollary I: The Basic equation of the gravitation

We shall quote Newton's Proposition from Motte's translation (1729):

> In a space void of resistance, if a body revolves in any orbit along an immoveable centre, and in the least time describes any arc just then nascent; and the versed sine of that arc is supposed to be drawn, bisecting the chord, and produced passing through the centre of force: the centripetal force in the middle of the arc, will be as the versed sine directly and the square of the time inversely.

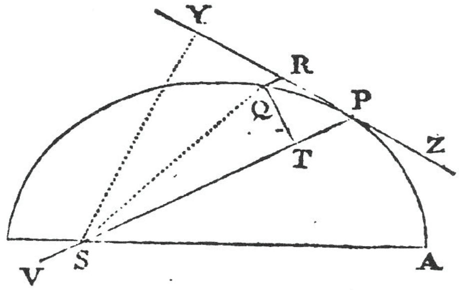


Fig.19. Proposition VI Theorem I in Book I of Principia:
the figure is taken from Principia, the third edition (1726).

Proof.

We suppose a celestial body revolving on the orbit and the centre, S, of the centripetal force located inside the orbit. Let P and Q be the present and next places of the body on the orbit; let T be the perpendicular foot from Q on SP; let R be the intersection of the tangent at P and the line drawn from Q and parallel to SP.

The product of SP and QT is in proportion to the area that the line joining the Sun and the body sweeps in a given time. Due to Kepler's second law this area is in proportion to the corresponding time, and hence $SP \times QT$ is in proportion to the time interval. The loss of altitude in this time interval is equal to QR. The acceleration is in proportion to the quotient of the travelling distance QR upon square of the time interval $SP \times QT$. Therefore the centripetal force is reciprocal to

$$SP^2 \times QT^2/QR, \tag{18}$$

where QR is also called 'versed sine.'

This is the basic equation of the universal gravitation given by Corollary I of this Proposition.

[Q.E.D.]

In the following subsections we shall show

$SP^2 \times QT^2/QR \to \text{constant} \times SP^2$

for elliptic, hyperbolic and parabolic orbits, as $Q \to P$.

3.2. Proposition XI Problem VI: Elliptic orbits

We shall quote Newton's Proposition from Motte's translation (1729):

> If a body revolves in an ellipsis: it is required to find the law of the centripetal force tending to the focus of the ellipsis.

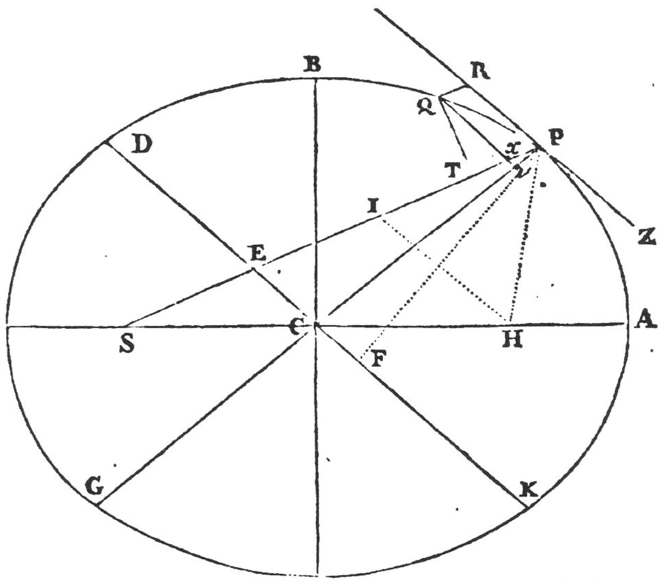


Fig.20. Proposition XI Problem VI in Book I of Principia: the figure is taken from Principia, the third edition (1726).

Proof.

Let P and Q be the present and next places of the body on the orbit; let PG, DK, S and H be a pair of the conjugate diameters and two foci, respectively; the centre of the centripetal force is S; let A and B be the vertices on the major and minor axes, respectively; draw QR parallel and QT perpendicular to SP; draw a line from Q parallel to RPz, and we have x and v, the intersections with SP and PG, respectively ; let E and F be the intersection of DK and SP and the perpendicular foot from P to DK; I is defined by the segment IH parallel to DK. We confirm the following lines are parallel: RPz // Qv // IH // DK, while QR // Px.

First we shall show $EP = AC$.

Since $SC = CH$, $ES = EI$ due to a counterproposition of the Triangle Midpoint Theorem applied to ΔSHI; $\angle RPS = \angle zPH$ and RPz // IH leads to the result, $\angle PIH = \angle PHI$; this implies ΔPHI is an isosceles triangle; hence $PI = PH$. Therefore

$$\begin{aligned} EP &= PS - ES \\ &= PS - \frac{PS - PI}{2} \\ &= \frac{PS + PI}{2} \\ &= \frac{PS + PH}{2} \\ &= AC, \end{aligned} \tag{19}$$

because $PS + PH = 2AC$ by the definition of ellipses.

Next we introduce the latus rectum. By use of (2)

$$L = 2BC^2/AC. \tag{20}$$

Starting from the identity,

$$\begin{aligned} L \cdot QR : L \cdot Pv &= QR : Pv \\ &= Px : Pv \\ &= EP : PC \end{aligned} \tag{21}$$

because of the similarity between ΔPxv and ΔPEC.

Due to the Power of a Point Theorem (3)

$$Pv \cdot vG : Qv^2 = PC^2 : CD^2.$$

Since $\angle QxT = \angle PEF$ as well as $\angle QTx = \angle PFE$, $\Delta QxT \sim \Delta PEF$ due to the AAA Theorem. Hence

$$\begin{aligned} Qx^2 : QT^2 &= EP^2 : PF^2 \\ &= AC^2 : PF^2 \\ &= CD^2 : BC^2 \end{aligned}$$

because of our Theorem I, Newton's Lemma XII in Book I of Principia: $AC \cdot BC = CD \cdot PF$; noting

$$Qv^2 : Qx^2 \to 1 : 1 \text{ as } Q \to P,$$

we obtain

$$Qv^2 : QT^2 \to Qx^2 : QT^2 = CD^2 : BC^2 \text{ as } Q \to P. \tag{22}$$

Now that our preparation is over, and let us consider the ratio of $L \cdot QR$ to QT^2:

$$
\begin{aligned}
L \cdot QR : QT^2 &\to L \cdot Pv \cdot \frac{EP}{PC} : Qv^2 \frac{BC^2}{CD^2} && (\because (21) \text{ and } (22)) \\
&= L \cdot Pv \cdot \frac{AC}{PC} : Pv \cdot vG \frac{CD^2}{PC^2} \frac{BC^2}{CD^2} && (\because (19) \text{ and } (3)) \\
&= \frac{L \cdot AC}{PC} : vG \frac{BC^2}{PC^2} && (\text{devided by } Pv) \\
&= L \cdot AC \cdot PC : vG \cdot BC^2 && (\text{multiplied by } PC^2) \\
&= 2\frac{BC^2}{AC} AC \cdot PC : vG \cdot BC^2 && (\because (20)) \\
&= 2PC : vG && (\text{devided by } BC^2) \\
&\to 2PC : PG \\
&= 1:1,
\end{aligned}
$$

as $Q \to P$.

Therefore

$$\text{The centripetal force } \propto \left(\frac{SP^2 \times QT^2}{QR}\right)^{-1} \to \left(L \times SP^2\right)^{-1},$$

as $Q \to P$. Since L is a constant, the centripetal force is found to be reciprocal to the square of *SP*, the distance to the focus, in case of the elliptic orbits.

[Q.E.I.]

3.3. Proposition XII Problem VII: Hyperbolic orbits

We shall quote Newton's Proposition from Motte's translation (1729):

> Suppose a body to move in an [*sic*] hyperbola: it is required to find the law of the centripetal force tending to the focus of that figure.

Proof.

Let *P* and *Q* be the present and next places of the body on the orbit; let *PG*, *DK*, *S* and *H* be a pair of the conjugate diameters and two foci, respectively; the centre of the centripetal force is *S*; let *A* and *B* be the positive principal vertices; draw *QR* parallel and *QT* perpendicular to *SP*; draw a line from *Q* parallel to *RPz*, and we have *x* and *v*, the intersections with *SP* and *PG*, respectively ; let *E* and *F* be the intersection of the extensions of *DK* and *SP* and the perpendicular foot from *P* to *DK*; *I* is defined by the intersection of a line from *H* parallel to *DK* and the extension of *SP*. We confirm the following lines are parallel: *Qv* // *RPz* // *DK* // *IH*, while *QR* // *Px*.

First we shall show $EP = AC$.

Since $HC = SC$, $IE = ES$ due to a counterproposition of the Triangle Midpoint Theorem applied to ΔSHI; $\angle IPz = \angle HPR$ due to Corollary of Theorem III and hence $\angle PIH = \angle PHI$; this implies ΔPHI is an isosceles triangle; hence $PI = PH$. Therefore

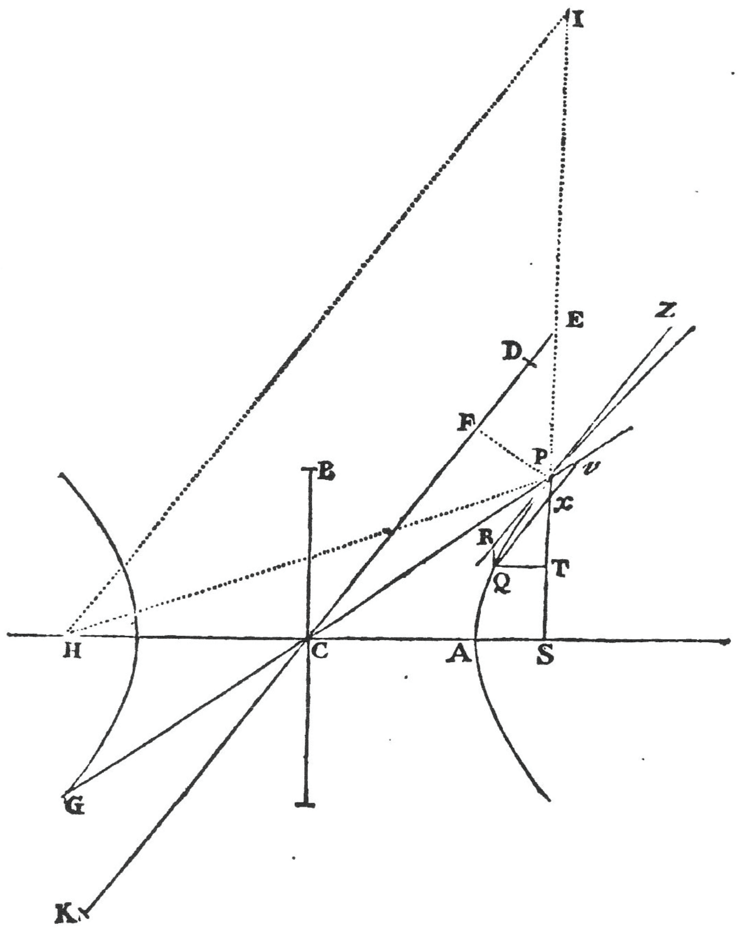


Fig.21. Proposition XII Problem VII in Book I of Principia
the figure is taken from Principia, third Edition (1726).

$$
\begin{aligned}
EP &= ES - SP \\
&= \frac{IS}{2} - SP \\
&= \frac{IS - SP - SP}{2} \\
&= \frac{IP - SP}{2} \\
&= \frac{HP - SP}{2} \\
&= AC,
\end{aligned}
\tag{23}
$$

because $HP - SP = 2AC$ by the definition of hyperbolas.

Next we introduce the principal latus rectum. By use of (9)

$$L = 2BC^2/AC. \tag{24}$$

Starting from the identity,

$$\begin{aligned} L \cdot QR : L \cdot Pv &= QR : Pv \\ &= Px : Pv \\ &= EP : PC \end{aligned} \tag{25}$$

because of the similarity between ΔPxv and ΔPEC.

Due to the Power of a Point Theorem (3)

$$Pv \cdot vG : Qv^2 = PC^2 : CD^2.$$

Since $\angle QxT = \angle PEF$ as well as $\angle QTx = \angle PFE$, $\Delta QxT \sim \Delta PEF$ due to the AAA Theorem. Hence

$$\begin{aligned} Qx^2 : QT^2 &= EP^2 : PF^2 \\ &= AC^2 : PF^2 \\ &= CD^2 : BC^2 \end{aligned} \tag{26}$$

because of our Theorem I, Newton's Lemma XII in Book I of Principia: $AC \cdot BC = CD \cdot PF$.

As $Q \to P$,

$$Qv^2 : Qx^2 \to 1 : 1. \tag{27}$$

Now that our preparation is over, and let us consider the ratio of $L \cdot QR$ to QT^2:

$$\begin{aligned} L \cdot QR : QT^2 &= L \cdot Pv \cdot \frac{EP}{PC} : Qx^2 \frac{BC^2}{CD^2} && (\because (25) \text{ and } (26)) \\ &\to L \cdot Pv \cdot \frac{EP}{PC} : Qv^2 \frac{BC^2}{CD^2} && (\because (27)) \\ &= L \cdot Pv \cdot \frac{AC}{PC} : Pv \cdot vG \frac{CD^2}{PC^2} \frac{BC^2}{CD^2} && (\because (23) \text{ and } (3)) \\ &= \frac{L \cdot AC}{PC} : vG \frac{BC^2}{PC^2} && (\text{devided by } Pv) \\ &= L \cdot AC \cdot PC : vG \cdot BC^2 && (\text{multiplied by } PC^2) \\ &= 2\frac{BC^2}{AC} AC \cdot PC : vG \cdot BC^2 && (\because (24)) \\ &= 2PC : vG && (\text{devided by } BC^2) \end{aligned}$$

$$\rightarrow 2PC : PG$$
$$= 1:1,$$

as $Q \rightarrow P$.

Therefore

$$\text{The centripetal force } \propto \left(\frac{SP^2 \times QT^2}{QR}\right)^{-1} \rightarrow \left(L \times SP^2\right)^{-1},$$

as $Q \rightarrow P$. Since L is a constant, the centripetal force is found to be reciprocal to the square of SP, the distance to the focus, in case of the hyperbolic orbits.

[Q.E.I.]

3.4. Proposition XIII Problem VIII: Parabolic orbits

We shall quote Newton's Proposition from Motte's translation (1729):

> If a body moves in the perimeter of a parabola: it is required to find the law of the centripetal force tending to the focus of that figure.

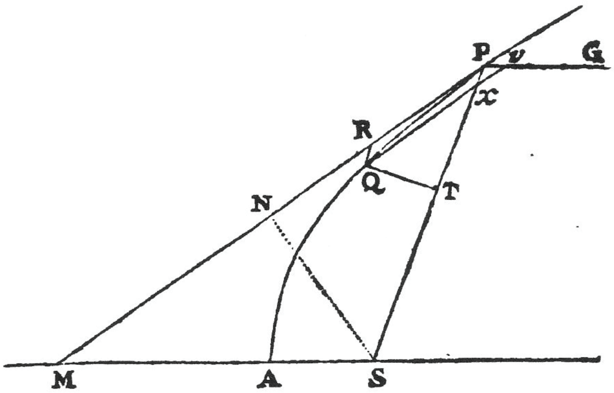


Fig.22. Proposition XIII Problem XIII in Book I of Principia: the figure is taken from Principia, the third edition (1726).

Proof.

Let P and Q be the present and next places of the body on the orbit; let PG and S be a diameter and the focus, respectively; the centre of the centripetal force is S; let A and M be the principal vertex and the intersection of the tangent to P and the extension of the principal diameter; draw QR parallel and QT perpendicular to SP; draw a line from Q parallel to the tangent to P, and we have x and v, the intersections with SP and PG, respectively ; let N be the perpendicular foot from S to the tangent. We confirm the following lines are parallel: MP // Qv; PG // MS; QR // SP.

Since $\angle Pvx = \angle PMS$ as well as $\angle vPx = \angle MSP$, $\Delta Pxv \sim \Delta SPM$ due to the AAA Theorem; due to our Theorem IX ΔSPM is an isosceles triangle, and hence

$Pv = Px$.
Further more
$Px = QR$,
because $MP \,//\, Qx$ and $QR \,//\, SP$. Therefore
$Pv = QR$.
Due to Theorem VII (17) holds. Substituting QR for Pv in (17), we obtain
$Qv^2 = 4SP \times QR$.
As $Q \to P$,
$Qv^2 : Qx^2 \to 1 : 1$.
Therefore

$$Qx^2 \to 4SP \times QR, \tag{28}$$

as $Q \to P$.

On the other hand $\Delta QxT \sim \Delta SPN$ due to the AAA Theorem, because $\angle QxT = \angle SPN$ as well as $\angle QTx = \angle SNP$; hence

$$\begin{aligned} Qx^2 : QT^2 &= SP^2 : SN^2 \\ &= SP : SA \qquad (\because (14)) \qquad (29) \\ &= 4SP \times QR : 4SA \times QR \qquad (\text{multiplied by } 4QR) \end{aligned}$$

Due to (28) and (29)

$$QT^2 \to 4SA \times QR, \tag{30}$$

as $Q \to P$. By Definition III $SA = p$, and hence $4SA = L$, the length of the principal latus rectum.

Therefore

$$\text{The centripetal force } \propto \left(\frac{SP^2 \times QT^2}{QR} \right)^{-1} \to \left(L \times SP^2 \right)^{-1},$$

as $Q \to P$. Since L is a constant, the centripetal force is found to be reciprocal to the square of SP, the distance to the focus, in case of the parabolic orbits.

[Q.E.I.]

4.CONCLUSION

We show the way to tackle Newton's Principia without Conics by Apollonius of Perga. We make most of symmetry in preparing our lemmas and theorems. Now the layperson can read through the heart of Principia just like Newton's contemporaries.

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