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Newton's Principia Revisited: A New Passage of the Solution to the Direct Kepler Problem

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Abstract

The direct Kepler problem, which Newton solved for the first time in the human history, is the task to induce physics behind Kepler's first and second laws. Newton broke through the problem by use of classical geometry and novel limiting operations. In this note by use of vector analysis a new passage of the solution starts from Kepler's first law, by way of conservation of the kinematics on the orbits, and arrives at the inverse-square law of gravitation as well as Kepler's second law at the same time. Kepler's first law is revealed essential to generate everything. Kepler's third law, derivative in its nature, is also confirmed within the same formalism.

Keywords: *Conic Sections*, *Celestial Mechanics*, *Kepler's Laws*, *Law of Gravitation*

1 Introduction

Liberal arts were the middle age curricula: these consist of two categories, the trivium and the quadrivium. The trivium, meaning 'three ways', are grammar, logic and rhetoric, i.e., arts of communications. The quadrivium, meaning 'four ways', are arithmetic, geometry, music and astronomy. Music is regarded as dynamic arithmetic, whilst astronomy is dynamic geometry.

Isaac Newton solved the direct Kepler problem [1]-[4]. This is the assignment to induce the physical principle out of Kepler's first and second laws [5] & [6], i.e., laws of ellipses and areal velocity. The task requires proficient knowledge about the conic sections and is even today much harder than solving the differential equation of gravitation to obtain the conic orbits notably done in a perfect way for the first time by Leonhard Euler.

We revisit Kepler's laws first. He introduced the first and second laws in [5]. In [6] he refined the first law and introduced newly the third law.

- The first law: the orbit of a planet is elliptical, and the Sun, the faucet of motion, is at one of the foci of the ellipse.
- The second law: a sectorial area is a measure of time.
- The third law: the ratio between the periodic times of any two planets is precisely the ratio of 3/2th power of their mean distances.

These are the three laws by Kepler's words. Newton posed the problem by stating 'a centripetal force towards a focus', and he used the sectorial area instead of time. The word 'centripetal' was coined by Newton at this moment. Newton extended the law applicable to all the three kinds of the conic sections. The parabolic orbit paved the way to the theory of comets.

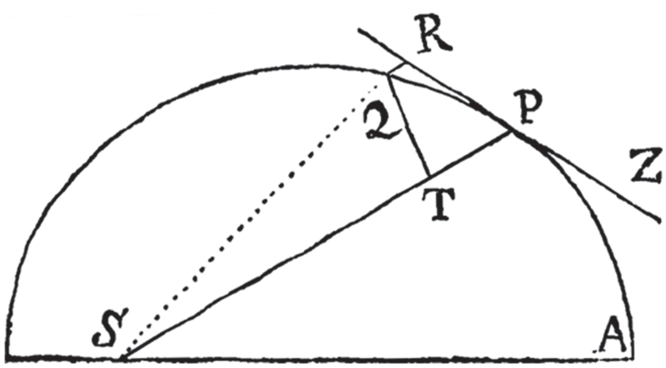

Figure 1. Newton's Strategy to tackle at the direct Kepler problem.
This figure is adjoined to Prop.VI Theor.V in the first book of Principia.

We shall review Newton's method of solution. Figure 1 shows geometry essential for deriving the law of gravitation. Without the centripetal force the planet at P passes along the tangent towards R, but actually the planet is pulled back onto the orbit at Q by the centripetal force acting parallel to *SP*. Accordingly the drop in the altitude *QR* is parallel to *SP*. So far Newton uses Kepler's first law. Drop the perpendicular from *Q* upon *SP*, and let *T* be its foot. Then the area *QT*×*SP* is proportional to time by Kepler's second law, and hence a centripetal acceleration is proportional to $QR/(QT \times SP)^2$. This is the conclusion of Prop.VI Theor.V in Principia. By use of theorems on the conic sections Newton shows

$$\frac{QR}{(QT \times SP)^2} \to \frac{1}{L \times SP^2} \quad \text{as} \quad Q \to P,$$

where *L* is the latus rectum of the conic section, *i.e.*, a length scale. Newton's method of solution is a combination of *classical* geometry and *novel* limiting operations.

Newton's Principia, in full name 'Philosophiae Naturalis Principia Mathematica', has been published three times [2]-[4], preceded by his lecture note [1]. All the editions are written in Latin, the global language at that time. In the summer of 1684 Lucasian Professor Newton wrote this lecture note in response to Edmond Halley's enquiry about the direct Kepler problem. In Cambridge Newton told Halley that he once solved the problem, but he could not find the script at this moment. The first edition of Principia was edited and published by Halley in 1687. Halley also drew all the figures but one (the orbit of the comet Kirch). Principia consists of three books: the first is written about the solution to the direct Kepler problem; the second is on fluid mechanics, which is Newton's rebuttal to Descartes' swirling vortex for the motion of the celestial bodies; the third is entitled 'the system of the world.' In the third book Newton explains real world problems: relation between the moon and tides, the moon's theory and so forth. He used data about the comet Kirch in 1680 to demonstrate the validity of his theory: the determined parabolic orbit is engraved by etching in a separate sheet and inserted into the book (Fig. 2). Even a randomly appearing comet obeys the law of gravitation.

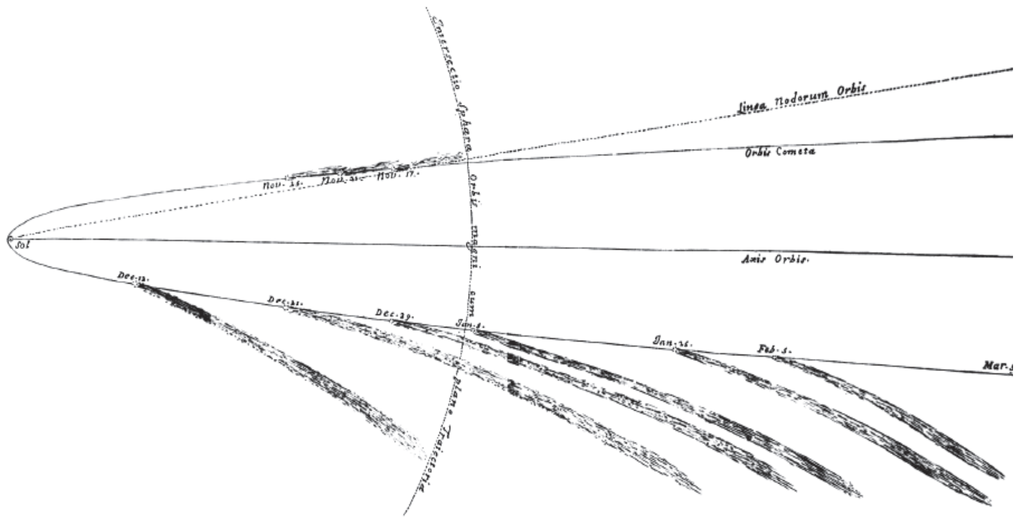

Figure 2. The Orbit of the comet Kirch: reproduced from the first edition of Principia (1687). Arc is the orbit of the earth.

In 1713 the second edition was edited by Roger Cotes (famous for the Newton-Cotes quadrature of numerical integration) and published by Cambridge University Press. Most of errors are corrected in this edition. At that time Newton was not a Professor at Cambridge and worked for the Royal Mint in London. The third edition was edited by Henry Pemberton and published by the Royal Society in 1726. Next year Newton died.

Principia was retold by use of modern mathematical formalism notably by Chandrasekhar [7]. Chandrasekhar's annotation is mainly made by use of algebraic geometry to confirm Newton's verbal derivation, but important theorems about the conic sections are introduced without reasoning those bases. Bruce [8] is a respectful piece, but he started the story from the inverse-square law towards the conic orbits. That derivation is not what Newton did in Principia.

We would like to add a new aspect to an old problem. The aim of this note is to present a novel method of solution to the direct Kepler problem in a concise and self-contained manner. Appendix introduces Newton's speculation on the inverse-square law [9] as an attempt in vein.

2 Theory

We describe the problem in three dimensional space and time. Without losing generality we suppose the celestial body orbits around the sun in *xy*-plane.

Definitions

$\mathbf{r} = (x, y, 0)$: the position vector of the celestial body.
$\mathbf{v} = (\mathrm{d}x/\mathrm{d}t, \mathrm{d}y/\mathrm{d}t, 0)$: the velocity vector of the celestial body.
$\mathbf{a} = (\mathrm{d}^2x/\mathrm{d}t^2, \mathrm{d}^2y/\mathrm{d}t^2, 0)$: the acceleration vector of the celestial body.
$\mathbf{i} = \mathbf{r}/r = (x/r, y/r, 0)$: the unit position-vector of the celestial body, where $r = |\mathbf{r}|$.
$\mathbf{h} = \mathbf{r} \times \mathbf{v} = (0,0, x\mathrm{d}y/\mathrm{d}t - y\mathrm{d}x/\mathrm{d}t) = (0,0,h)$: the specific angular-momentum vector of the celestial body; let h be a non-zero z-component of $\mathbf{h}$; we assume the prograde rotation, that is h is positive.
$\mathbf{e} = (-e, 0,0)$: the eccentricity vector.

Remark

The specific angular-momentum vector $\mathbf{h}$ is orthogonal to all other vectors.

Lemma I*: nature of the unit position-vector*

$$\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t} = \frac{\mathbf{h} \times \mathbf{r}}{r^3} . \tag{1}$$

[Proof]

Direct calculation leads us to the result above with the aid of the vector triple product.

$$\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t} = \frac{1}{r}\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}t} + \mathbf{r}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{r}\right) = \frac{\mathbf{v}}{r} - \frac{\mathbf{r}(\mathbf{v}\cdot\mathbf{r})}{r^3} = \frac{\mathbf{v}(\mathbf{r}\cdot\mathbf{r}) - \mathbf{r}(\mathbf{v}\cdot\mathbf{r})}{r^3}$$

$$= \frac{\mathbf{r}\times(\mathbf{v}\times\mathbf{r})}{r^3} = \frac{(\mathbf{r}\times\mathbf{v})\times\mathbf{r}}{r^3} = \frac{\mathbf{h}\times\mathbf{r}}{r^3}.$$

[Q.E.D.]

Corollary I

$$\mathbf{r}\cdot\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t} = 0. \tag{2}$$

[Proof]

We show in the following:

$$\mathbf{r}\cdot\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t} = \mathbf{r}\cdot\frac{\mathbf{h}\times\mathbf{r}}{r^3} = \frac{\mathbf{h}\cdot(\mathbf{r}\times\mathbf{r})}{r^3} = 0.$$

[Q.E.D.]

Corollary II

$$\mathbf{v}\cdot\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t} = \frac{h^2}{r^3}. \tag{3}$$

[Proof]

We obtain the following result:

$$\mathbf{v}\cdot\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t} = \mathbf{v}\cdot\frac{\mathbf{h}\times\mathbf{r}}{r^3} = \frac{\mathbf{h}\cdot(\mathbf{r}\times\mathbf{v})}{r^3} = \frac{\mathbf{h}\cdot\mathbf{h}}{r^3} = \frac{h^2}{r^3}.$$

[Q.E.D.]

Theorem I: *Kepler's first law extended to all the three kinds of the conic sections*

Extended Kepler's first law is given by words:

Celestial bodies orbit along the conic sections with the Sun at one of the foci.

In case of a parabola there is only one focus. Extending applicability of the law to all the conics, we adopt the vector equation of the conic sections:

$$\mathbf{r}\cdot(\mathbf{e}+\mathbf{i}) = \frac{L}{2}, \tag{4}$$

where L is the *latus rectum* of the conic section. The latus rectum is a parameter, which has the length of the chord parallel to the *directrix* and running through the focus.

[Proof]

Figure 3 shows defining the conic section by use of the directrix at $x = -d$ and the focus at the origin O. By the definition of the eccentricity $e = OP/QP$, and hence we have

$$e = \frac{r}{x+d}.$$

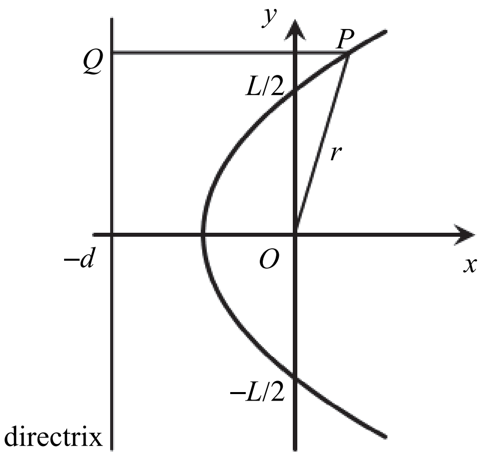

Figure 3. Definition of the conic section

The planet P orbits on the conic section. The Sun sits on the focus O. Q is the foot of the perpendicular from P to the directrix running at $x = -d$.

In case $x = 0$, r is equal to $L/2$. Therefore ed is found to be equal to $L/2$. Accordingly we obtain

$$ex + \frac{L}{2} = r,$$

or

$$-ex + r = \frac{L}{2},$$

that is

$$\mathbf{r} \cdot (\mathbf{e} + \mathbf{i}) = \frac{L}{2}.$$

[Q.E.D.]

***Corollary III**: the equation of the velocity vector*

The velocity vector satisfies the following equation.

$$\mathbf{v} \cdot (\mathbf{e} + \mathbf{i}) = 0. \tag{5}$$

[Proof]

Differentiating eqn (4) with respect to time, we obtain

$$\mathbf{v}\cdot(\mathbf{e}+\mathbf{i})+\mathbf{r}\cdot\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t}=0.$$

But the second term is naught because of eqn (2), and hence we obtain the required result.

[Q.E.D.]

Corollary IV*: the equation of the acceleration vector*

The acceleration vector satisfies the following equation.

$$\mathbf{a}\cdot(\mathbf{e}+\mathbf{i})=-\frac{h^2}{r^3}. \tag{6}$$

[Proof]

Differentiating eqn (5) with respect to time, we obtain

$$\mathbf{a}\cdot(\mathbf{e}+\mathbf{i})+\mathbf{v}\cdot\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t}=0.$$

But the second term is equal to h^2/r^3 because of eqn (3), and hence we obtain the required result.

At this stage the orientation of **a** is not specified, and h is not necessarily constant.

[Q.E.D.]

Theorem II*: Conservation of kinematics on the orbits, or the law of gravitation in Newton's form*

We shall show that conservation of kinematics on the orbits leads us to the law of gravitation in Newton's form.

In the direct Kepler problem the orbits do not temporally change their shapes. On the orbit there is the same motion at the same position at any time. Therefore the kinematics on the orbits is conserved.

We deduce the following identity from eqn (5).

$$\mathbf{v}\times\mathbf{h}=\frac{2h^2}{L}(\mathbf{e}+\mathbf{i}). \tag{7}$$

The left-hand side is the vector product between translational and angular momenta, whilst the right-hand side is a combination of the conic parameters and a position. Therefore the identity (7) is a summary of kinematics. Enforcing time-independence upon the kinematic identity (7), we simultaneously obtain

$$\mathbf{a}=-\frac{2h^2}{L}\frac{\mathbf{i}}{r^2}, \tag{8}$$

the inverse-square law of gravitation, satisfying eqn (6), and

$$\frac{\mathrm{d}h}{\mathrm{d}t}=0, \tag{9}$$

Kepler's second law: h, corresponding to twice a sectorial area, is constant.

[Proof]

Due to eqn (5), **v** is perpendicular to **e** + **i**. As is stated in *Remark*, **h** is perpendicular to both **v** and **e** + **i**. Therefore **v**, **h** and **e** + **i** are perpendicular to one another, and hence (see Fig.4)

$$\mathbf{v}\times\mathbf{h}\parallel\mathbf{e}+\mathbf{i}.$$

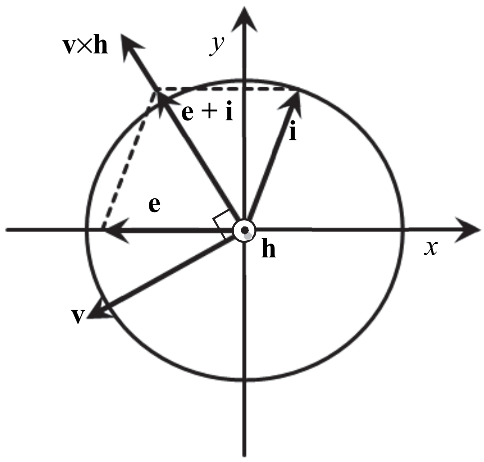

Figure 4. Relations of **v**, **h**, **i** and **e** vectors (a generic image): a celestial body is thought to orbit in a prograde manner, and hence **h** points upwards in the positive *z*-direction. The vector **i** is on a unit circle.

The following is a straight forward calculation:

$$\mathbf{r}\cdot(\mathbf{v}\times\mathbf{h}) = \mathbf{h}\cdot(\mathbf{r}\times\mathbf{v}) = \mathbf{h}\cdot\mathbf{h} = h^2.$$

Comparing the result above with eqn (4), we establish the identity (7):

$$\mathbf{v}\times\mathbf{h} = \frac{2h^2}{L}(\mathbf{e}+\mathbf{i}).$$

To demonstrate the required result we use eqn (1) and the following relation.

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \mathbf{h}h^{-1}\frac{\mathrm{d}h}{\mathrm{d}t},$$

because **h** has *z*-component only.

Let us differentiate the kinematic identity (7) with respect to time.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left\{\mathbf{v}\times\mathbf{h} - \frac{2h^2}{L}(\mathbf{e}+\mathbf{i})\right\} &= \mathbf{a}\times\mathbf{h} + \mathbf{v}\times\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} - \frac{4h}{L}\frac{\mathrm{d}h}{\mathrm{d}t}(\mathbf{e}+\mathbf{i}) - \frac{2h^2}{L}\frac{\mathrm{d}\mathbf{i}}{\mathrm{d}t}\\ &= \mathbf{a}\times\mathbf{h} + \mathbf{v}\times\mathbf{h}h^{-1}\frac{\mathrm{d}h}{\mathrm{d}t} - \frac{4h}{L}\frac{\mathrm{d}h}{\mathrm{d}t}(\mathbf{e}+\mathbf{i}) - \frac{2h^2}{L}\frac{\mathbf{h}\times\mathbf{r}}{r^3}\\ &= \mathbf{a}\times\mathbf{h} + \frac{2h^2}{L}(\mathbf{e}+\mathbf{i})h^{-1}\frac{\mathrm{d}h}{\mathrm{d}t} - \frac{4h}{L}\frac{\mathrm{d}h}{\mathrm{d}t}(\mathbf{e}+\mathbf{i}) + \frac{2h^2}{L}\frac{\mathbf{r}\times\mathbf{h}}{r^3}\\ &= \left(\mathbf{a} + \frac{2h^2}{L}\frac{\mathbf{i}}{r^2}\right)\times\mathbf{h} - \frac{2h}{L}\frac{\mathrm{d}h}{\mathrm{d}t}(\mathbf{e}+\mathbf{i}). \qquad (10)\end{aligned}$$

Vectors **a** and **i** are not perpendicular to **e** + **i** because of eqns (4) and (6). Therefore, as shown in Fig.5, the first term on the right-hand side of eqn (10) above never becomes parallel to **e** + **i**, the second term. But the time derivative has to vanish. There is the one and only possibility that both terms are zero vectors. This is the case.

We obtain the conditions of making both terms into zero vectors in eqn (10) *i.e.*,

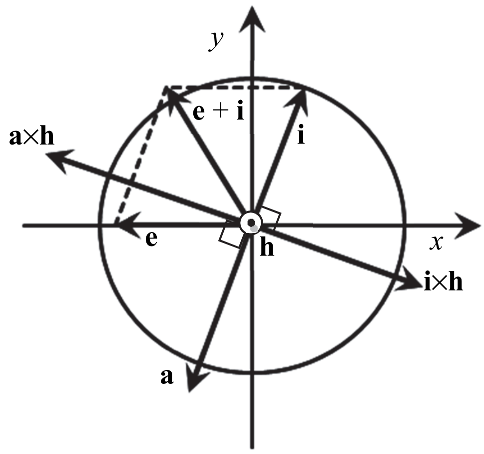

Figure 5. Relations of **a**, **h**, **i** and **e** vectors (a generic image): **h** points upwards in the positive *z*-direction; the orientation of **a** is imaginary.

$$\mathbf{a} = -\frac{2h^2}{L}\frac{\mathbf{i}}{r^2}$$

for the first term as well as

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 0$$

for the second term.

The equivalence of the two propositions above comes from the following identity.

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}t} = \frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{r}\times\mathbf{v}) = \mathbf{v}\times\mathbf{v} + \mathbf{r}\times\mathbf{a} = r\mathbf{i}\times\mathbf{a}. \tag{11}$$

Therefore $\mathrm{d}h/\mathrm{d}t = 0$, if $\mathbf{a} \parallel \mathbf{i}$; $\mathbf{a} \parallel \mathbf{i}$, if $\mathrm{d}h/\mathrm{d}t = 0$; this is another sense of Kepler's second law.

Then we confirm eqn (6) by use of eqn (4) as follows.

$$\mathbf{a}\cdot(\mathbf{e}+\mathbf{i}) = -\frac{2h^2}{L}\frac{\mathbf{i}}{r^2}\cdot(\mathbf{e}+\mathbf{i}) = -\frac{2h^2}{L}\frac{1}{r^3}\mathbf{r}\cdot(\mathbf{e}+\mathbf{i}) = -\frac{h^2}{r^3}.$$

[Q.E.D.]

Corollary V: *Laplace-Runge-Lenz vector*

The vector below is called the Laplace-Runge-Lenz vector and conservative.

$$\mathbf{v}\times\mathbf{h} - \frac{2h^2}{L}\mathbf{i} \tag{12}$$

This vector is a summary of the kinematics on the orbit.

[Proof]

Now *h* is found to be constant. Rewriting the kinematic identity (7), we obtain

$$\mathbf{v}\times\mathbf{h} - \frac{2h^2}{L}\mathbf{i} = \frac{2h^2}{L}\mathbf{e}.$$

Therefore the Laplace-Runge-Lenz vector is a constant vector, and hence

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{v}\times\mathbf{h}-\frac{2h^2}{L}\mathbf{i}\right)=\mathbf{0}.$$

[Q.E.D.]

Scholium

The proportional constant of the inverse-square acceleration, $2h^2/L$, is universal to the particular celestial system. Every orbiting celestial member has different h, different L but the same $2h^2/L$ in this system. Newton states this proposition as Prop.XIV Theor.VI in the first book of Principia: L is proportional to h^2. As for the solar system $2h^2/L$ is equal to $GM_{\odot}$, where G and $M_{\odot}$ designate the universal constant of gravitation and the mass of the Sun, respectively.

***Theorem III**: Kepler's third law*

The ratio between the periodic times of any two planets is precisely the ratio of the 3/2th power of the semi-major axes.

[Proof]

Suppose the elliptic orbit has a and b as its semi-major and semi-minor axes, respectively. Then the area of the ellipse is equal to πab, and the latus rectum L is equal to $2b^2/a$.

We start from the identity. Let T be the periodic time. Noting h is twice a sectorial area, we obtain

$$hT=\int_0^T h\mathrm{d}t=2\pi ab=2\pi a\sqrt{aL/2}=\pi\sqrt{2L}a^{3/2}.$$

Therefore

$$\frac{T}{a^{3/2}}=2\pi\left(\frac{2h^2}{L}\right)^{-1/2}, \tag{13}$$

for all the celestial bodies in this system because of the universality of $2h^2/L$ as stated in *Scholium* of *Theorem II* above.

[Q.E.D.]

3 Conclusion

We, fortified by vector analysis, can solve the direct Kepler problem in a concise and self-contained manner. There is a new passage: Kepler's first law generates equations of the orbit, the velocity and the acceleration; accordingly the kinematic identity on the orbit is deduced; finally conservation of the kinematics* leads us to both the inverse-square law of gravitation and Kepler's second law; the Laplace-Runge-Lenz vector comes out as a by-product. Kepler's third law is confirmed within the same formalism.

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* A reviewer points out that conservation of the kinematics on the conic orbits should be added to the basic laws of the direct Kepler problem.

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Appendix: Newton's speculation of the inverse-square law (c1669)

There is a sheet of Newton's manuscript [9] that he derives the formula about the acceleration caused by the constant circular motion, *i.e.* in conventional form, $r\omega^2$, where ω denotes the angular velocity. It is the same one that Huygens deduced for the first time in the human history. In the end of this manuscript Newton says (originally in Latin)

> Finally, among the primary planets, since the cubes of their distances from the Sun are reciprocally as the squared numbers of their periods in a given time, their endeavours of recess from the Sun will be reciprocally as the squares to their distance from the Sun ···.

Due to Kepler's third law ω is proportional to $r^{-3/2}$, and hence the centripetal acceleration becomes proportional to r^{-2}. Newton's contemporary reached the same conclusion along the same logic by use of Kepler's third law, but this rule of thumb did not lead anyone to fruitful conclusions. Kepler's third law is deduced from integrated values, so there is no passage to dynamics.