Far East Journal of Applied Mathematics
Volume …, Number …, 2008, Pages …
This paper is available online at http://www.pphmj.com

ANALYSIS ON THE NATURE OF THE BASIC EQUATIONS IN SYNERGETIC INTER-REPRESENTATION NETWORK

TAKESHI SUGIMOTO

Faculty of Engineering
Kanagawa University
Kanagawa Ward
Yokohama, 221-8686, Japan
e-mail: take@is.kanagawa-u.ac.jp

Abstract

Synergetic Inter-Representation Network is described by n-dimensional ordinary differential equations with cubic nonlinearity. The novel find is the fact that the basic equation set is transformed to a set of n-dimensional Lotka-Volterra equations. The existence and stability of fixed points is shown mathematically rigorously by a series of inequality conditions. For the sake of self-consistency and numerical efficiency, it is proposed to use the n-dimensional Lotka-Volterra equations having n stable fixed points.

1. Introduction

Synergetic Inter-Representation Network (SIRN in short) is the paradigm for human cognitive process that is first proposed as IRN by Portugali and then augmented by Synergetics (see for example [1, 2]). Their representations cover artifacts, natural entities, behaviors and anything in our mind [2]. Its mathematical model is derived from the one for Synergetic pattern recognition [1]. The state variable vector is

2000 Mathematics Subject Classification: 37F99, 91E10, 34D45.

Keywords and phrases: synergetics, inter-representation network, fixed points.

Received October 24, 2008

expandable by n representations in such a manner:

$$\mathbf{x} = \sum_{i=1}^{n} \xi_i \mathbf{v}_i, \tag{1}$$

where

$\mathbf{x}$: the state variable vector;

$\xi_i = \mathbf{v}_i^+ \cdot \mathbf{x}$: the order parameter for the ith representation;

$\mathbf{v}_i$: the pattern vector of the ith representation;

$\mathbf{v}_i^+$: its adjoint.

Its dynamics is described by an autonomous ordinary differential equation with respect to $\mathbf{x}$ with cubic nonlinearity:

$$\frac{d}{dt}\mathbf{x} = \sum_{i=1}^{n}\left(\lambda_i - B\sum_{j\neq 1}\xi_j^2\right)\xi_i\mathbf{v}_i - C|\mathbf{x}|^2\mathbf{x}, \tag{2}$$

where

λ_i : the growth rate called the *attention parameter* for ith representation;

B : the strength of competition among representations;

C : the parameter constraining the growth for all the representations.

All these parameters are positive constants.

Taking the inner product of (2) and $\mathbf{v}_i^+$ with the aid of (1), we obtain the governing equations of the order parameters:

$$\frac{d}{dt}\xi_i = \left(\lambda_i - \sum_{j=1}^{n} D_{ij}\xi_j^2\right)\xi_i, \tag{3}$$

where

$$D_{ij} = \begin{cases} C & i = j, \\ B + C & i \neq j. \end{cases}$$

2. Transformation to Lotka-Volterra Equations

Multiplying (3) by ξ_i and substituting η_i for ξ_i^2, we obtain

$$\frac{d}{dt}\eta_i = 2\left(\lambda_i - \sum_{j=1}^{n} D_{ij}\eta_j\right)\eta_i. \tag{4}$$

It is a set of n-dimensional Lotka-Volterra equations, having quadratic nonlinearity. We should note that the new variable η_i is non-negative.

3. Nature of the Fixed Points

The fixed points are obtained by solving

$$\lambda_i - \sum_{j=1}^{n} D_{ij}\eta_j = 0 \tag{5}$$

or

$$\eta_i = 0 \tag{6}$$

for $i = 1, 2, ..., n$. Thus, the choice is 2^n, which is equal to the total number of the fixed points. Let us denote $\boldsymbol{\eta}^{(k,\, l)}$ as the lth fixed point among those having k non-zero components; l is in $[1, {}_nC_k]$.

The first is the total extinction of all the representations:

$$\boldsymbol{\eta}^{(0,\, 1)} = (0, 0, ..., 0)^T.$$

The second is a set of n single-representation points:

$$\boldsymbol{\eta}^{(1,\, i)} = (0, ..., 0, \lambda_i / C, 0, ..., 0)^T$$

for $i = 1, 2, ..., n$.

In the rest of the fixed points, several representations coexist. For example

$$\boldsymbol{\eta}^{(k,\, 1)} = (\eta_1^{(k,1)}, ..., \eta_k^{(k,1)}, 0, ..., 0)^T.$$

Each component is obtained mathematically rigorously. Summing up (5) for $i = 1$ to k, we obtain

$$\{(k-1)B + kC\}\sum_{i=1}^{k} \eta_i^{(k,1)} = \sum_{i=1}^{k} \lambda_i. \tag{7}$$

The equations (5) and (7) lead us to the solution set:

$$\eta_i^{(k,1)} = \frac{1}{B}\left\{\frac{B+C}{(k-1)B+kC}\sum_{j=1}^{k}\lambda_j - \lambda_i\right\}. \tag{8}$$

If this value is negative, such a fixed point is not feasible.

Now, we shall show

$$\frac{B+C}{(k-1)B+kC}\sum_{j=1}^{k}\lambda_j > \lambda_i \tag{9}$$

for all the i under the condition of n-representations coexistence:

$$\frac{B+C}{(n-1)B+nC}\sum_{j=1}^{n}\lambda_j > \lambda_i \tag{10}$$

for all the i.

Let us start from the following inequality for $k < n$:

$$\frac{1}{(k-1)B+kC}\sum_{j=1}^{k}\lambda_j > \frac{1}{kB+(k+1)C}\sum_{j=1}^{k+1}\lambda_j. \tag{11}$$

This can be shown by straightforward algebra as follows:

$$\frac{1}{(k-1)B+kC}\sum_{j=1}^{k}\lambda_j - \frac{1}{kB+(k+1)C}\sum_{j=1}^{k+1}\lambda_j$$

$$= \left\{\frac{1}{(k-1)B+kC} - \frac{1}{kB+(k+1)C}\right\}\sum_{j=1}^{k}\lambda_j - \frac{\lambda_{k+1}}{kB+(k+1)C}$$

$$= \frac{1}{(k-1)B + kC}\left\{\frac{B + C}{kB + (k+1)C}\sum_{j=1}^{k}\lambda_j - \frac{(k-1)B + kC}{kB + (k+1)C}\lambda_{k+1}\right\}$$

$$= \frac{1}{(k-1)B + kC}\left\{\frac{B + C}{kB + (k+1)C}\sum_{j=1}^{k+1}\lambda_j - \lambda_{k+1}\right\}. \tag{12}$$

In case $k = n - 1$, the last line of (12) is positive because of (10); hence (11) holds true; then mathematical induction leads us to the conclusion that (11) holds true for any k; then (9) holds true for any k because of (10) and (11).

The upper bound of (9) is given by

$$\frac{B + C}{C}\lambda_j > \lambda_i \tag{13}$$

for all the i and j.

Now, we shall check stability of the fixed points. We need to calculate the Jacobian of (4):

$$\mathbf{J} = \left\{\frac{\partial}{\partial \eta_j}\left(\frac{d\eta_i}{dt}\right)\right\}, \tag{14}$$

where

$$\frac{\partial}{\partial \eta_j}\left(\frac{d\eta_i}{dt}\right) = \begin{cases} 2\left\{\lambda_i - (B + C)\sum_{l=1}^{n}\eta_l + (B - C)\eta_i\right\} & j = i, \\ -2(B + C)\eta_i & j \neq i. \end{cases}$$

In case of $\boldsymbol{\eta}^{(0,1)}$,

$$\frac{\partial}{\partial \eta_j}\left(\frac{d\eta_i}{dt}\right) = \begin{cases} 2\lambda_i & j = i, \\ 0 & j \neq i. \end{cases}$$

Therefore, $\mathbf{J}$ is positive definite and hence $\boldsymbol{\eta}^{(0,1)}$ is an unstable node.

In case of $\boldsymbol{\eta}^{(1,k)}$,

$$\frac{\partial}{\partial \eta_j}\left(\frac{d\eta_i}{dt}\right) = \begin{cases} \begin{cases} -2\left(\dfrac{B+C}{C}\lambda_k - \lambda_i\right) & j = i \neq k, \\ -\lambda_k & j = i = k, \end{cases} \\ 0 \quad j \neq i. \end{cases}$$

Therefore, $\mathbf{J}$ is negative definite because of (13) and hence $\boldsymbol{\eta}^{(1,k)}$ for all the k are stable nodes.

In case of $\boldsymbol{\eta}^{(k,1)}$,

$$\frac{\partial}{\partial \eta_j}\left(\frac{d\eta_i}{dt}\right) = \begin{cases} \begin{cases} -2C\eta_i^{(k,1)} & j = i \text{ and } i \leq k, \\ -2\left(\dfrac{B+C}{(k-1)B+kC}\sum_{l=1}^{k}\lambda_l - \lambda_i\right) & j = i \text{ and } i \geq k+1, \end{cases} \\ \begin{cases} -2(B+C)\eta_i^{(k,1)} & j \neq i \text{ and } i \leq k, \\ 0 & j \neq i \text{ and } i \geq k+1. \end{cases} \end{cases}$$

This Jacobian assumes the following form:

$$\mathbf{J}(\boldsymbol{\eta}^{(k,1)}) = \begin{pmatrix} \mathbf{A}_{kU} & * \\ 0 & \mathbf{A}_{kL} \end{pmatrix},$$

where $\mathbf{A}_{kU}$ covers the components with $i = 1$ to k and $j = 1$ to k, while $\mathbf{A}_{kL}$ covers the components with $i = k+1$ to n and $j = k+1$ to n. Then the determinant of the Jacobian becomes as follows:

$$|\mathbf{J}(\boldsymbol{\eta}^{(k,1)})| = |\mathbf{A}_{kU}||\mathbf{A}_{kL}|.$$

The lower part $\mathbf{A}_{kL}$ is negative definite because this diagonal matrix has all negative components.

To check definiteness of $\mathbf{A}_{kU}$, we shall test it by using two particular vectors: $\mathbf{y}$ has $y_i = y_j = 1$ and zero for the rest; $\mathbf{z}$ has $z_i = -z_j = 1$ and zero for the rest. Then, we have

$$\mathbf{y}^T\mathbf{A}_{kU}\mathbf{y} = -2(B+2C)(\eta_i^{(k,1)} + \eta_j^{(k,1)}) < 0$$

and

$$\mathbf{z}^T\mathbf{A}_{kU}\mathbf{z} = 2B(\eta_i^{(k,1)} + \eta_j^{(k,1)}) > 0.$$

Therefore, $\mathbf{A}_{kU}$ is apparently indefinite. This in turn implies that $\mathbf{J}(\boldsymbol{\eta}^{(k,1)})$ is also indefinite and that this Jacobian has some positive and some negative eigenvalues; hence $\boldsymbol{\eta}^{(k,1)}$ is a saddle point. All such $2^n - n - 1$ points are found to be saddle points in the very same manner. In the vicinity of these saddle points, zero components will never grow again, while some of non-zero components will grow further. Thus, n representations are selected in this dynamical system and in the end only one survives.

4. Conclusions

To make most of the Lotka-Volterra formalism, we mathematically rigorously explore the nature of the fixed points.

Here is our proposal. As far as the state variable $\mathbf{x}$ is expanded by n representations $\mathbf{v}_i$ for $i = 1$ to n, the total number of the stable fixed points should be n. In this sense, the Lotka-Volterra formalism is more natural than the original formalism, which has $2n$ stable fixed points: these have n positive and n negative order parameters. The original formalism has cubic nonlinearity, and hence numerical scheme must have a fine time interval to accurately integrate steep changes. On the other hand, the Lotka-Volterra formalism has quadratic nonlinearity: the total number of multiplying operations is apparently reduced; a less fine time interval is allowed because of less steep changes.

Acknowledgement

I thank Professor Narita for encouraging discussion on this topic.

References

[1] H. Haken and J. Portugali, Synergetic Cities I and II, Self-Organization and the Cities, Springer, Heiderberg, (1999), pp. 262-302.

[2] J. Portugali, The Seven Basic Propositions of SIRN, Nonlinear Phenomena in Complex Systems 5(4) (2002), 428-444.

■